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Convolutional Filtering on Sampled Manifolds

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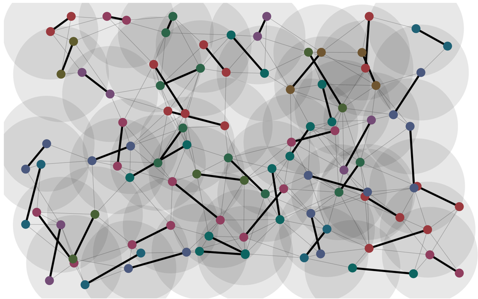
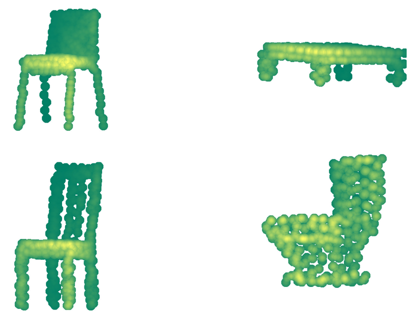
Filtering in Non-Euclidean Domains

- Applications involving geometric data have gained increasing attention
  - ⇒ E.g., wireless communication networks, point clouds for 3D models



Ad-hoc network



Point clouds

- Graph convolutional filtering and manifold convolutional filtering have become the prominent choices for non-Euclidean signal processing
- Convolutional filtering provides the fundamental block for constructing deep learning architectures which helps establish geometric deep learning

Graphs (Sampled Manifolds) can approximate Manifolds

Fact Graphs with well-defined limits can be sampled from a manifold

- We relate manifold convolutional filters with graph convolutional filters

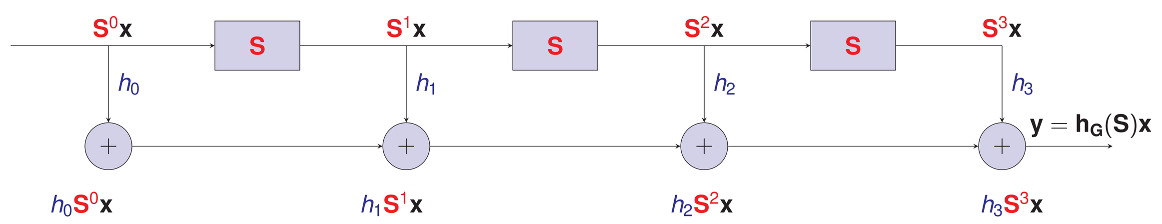
Our Contributions

- Construct convolutional filters on graphs sampled from the manifold
- Derive difference bounds between the graph Laplacian and Laplace-Beltrami operator from the operator and spectral aspects
- Show graph filtering converges to the manifold filtering with $n^{-1/(2d+8)}$
- Carry out experiments with navigation control and point cloud classification

Graph Signal Processing - Graph Filters

- Graph $\mathbf{G}$ with matrix $\mathbf{S}$ – graph shift operator – and graph signal $\mathbf{x} \in \mathbb{R}^n$
- Graph convolutional filter is defined as a summation of iterative graph shifts

$$\mathbf{y} = \mathbf{h}_{\mathbf{G}}(\mathbf{S})\mathbf{x} = \sum_{k=0}^{K-1} h_k \mathbf{S}^k \mathbf{x} \quad \text{– filter with coefficients } h_k$$



- The symmetric matrix $\mathbf{S}$ admits the eigenvector decomposition $\mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$

Spectral Representation of Graph Filters

Consider graph signal $\mathbf{x}$ and the filtered signal $\mathbf{y} = \sum_{k=0}^{K-1} h_k \mathbf{S}^k \mathbf{x}$. The Graph Fourier Transforms (GFTs) $\tilde{\mathbf{x}} = \mathbf{V}^H\mathbf{x}$ and $\tilde{\mathbf{y}} = \mathbf{V}^H\mathbf{y}$ are related by

$$\tilde{\mathbf{y}} = \sum_{k=0}^{K-1} h_k \mathbf{\Lambda}^k \tilde{\mathbf{x}} = \hat{h}(\mathbf{\Lambda})\tilde{\mathbf{x}} \quad \Rightarrow \quad \tilde{y}_i = \sum_{k=0}^{K-1} h_k \lambda_i^k \tilde{x}_i = \hat{h}(\lambda_i)\tilde{x}_i$$

- The graph filter frequency response is point-wise – $\hat{h}(\lambda) = \sum_{k=0}^{K-1} h_k \lambda^k$

Manifold Signal Processing – Manifold Filters

- d-dimensional manifold $\mathcal{M} \subset \mathbb{R}^N$ support manifold signals – $f \in L^2(\mathcal{M})$
- Laplace-Beltrami (LB) operator – intrinsic gradient and divergence

$$\mathcal{L}f = -\text{div}(\nabla f)$$

- Manifold convolutional filter is the integration of the heat diffusion dynamics

$$g(x) = (\mathbf{h}f)(x) = \int_0^\infty \tilde{h}(t)e^{-t\mathcal{L}}f(x)\text{d}t = \mathbf{h}(\mathcal{L})f(x)$$

- $\mathcal{L}$ is self-adjoint and positive semi-definite – a discrete spectrum $\{\lambda_i, \phi_i\}_{i\in\mathbb{N}^+}$

Spectral Representation of Manifold Filters

Consider manifold filter with impulse response $\tilde{h}(t)$, manifold signal $f(x)$ and the filtered signal $g(x) = \int_0^\infty \tilde{h}(t)e^{-t\mathcal{L}}\text{d}t f(x)$. The frequency components when projecting on the eigenfunctions $[\hat{f}]_i = \langle f, \phi_i\rangle_{L^2(\mathcal{M})}$ and $[\hat{g}]_i = \langle g, \phi_i\rangle_{L^2(\mathcal{M})}$ are related by

$$[\hat{g}]_i = \int_0^\infty \tilde{h}(t)e^{-t\lambda_i}\text{d}t[\hat{f}]_i = \hat{h}(\lambda_i)[\hat{f}]_i \qquad \Rightarrow \qquad g = \sum_{i=1}^\infty \hat{h}(\lambda_i)[\hat{f}]_i\phi_i$$

- The manifold filter frequency response is point-wise – $\hat{h}(\lambda) = \int_0^\infty \tilde{h}(t)e^{-t\lambda}\text{d}t$

Sampled Manifolds as Graphs

- Graphs on sampled points with geometric structure – sampled manifold
  - ⇒ $X = \{x_1, x_2, \dots, x_n\}$ are n discrete points sampled uniformly from $\mathcal{M}$
  - ⇒ The weight value connecting points x_i and x_j is set as a Gaussian kernel

$$w_{ij} = \frac{1}{n}\frac{1}{\epsilon(4\pi\epsilon)^{d/2}}\exp\left(-\frac{\|x_i - x_j\|^2}{4\epsilon}\right)$$

⇒ Adjacency matrix $[\mathbf{A}_n]_{ij} = w_{ij}$ ⇒ Laplacian matrix $\mathbf{L}_n^\epsilon = \text{diag}(\mathbf{A}_n\mathbf{1}) - \mathbf{A}_n$

Filtering on Sampled Manifolds

- Graph signal is a sampled manifold signal with a sampling operator $\mathbf{P}_n$

$$\mathbf{f} = \mathbf{P}_n f \text{ with } [\mathbf{f}]_i = f(x_i), \quad x_i \in X,$$

- Manifold filter can operate on the graph Laplacian in continuous time

$$\mathbf{g} = \int_0^\infty \tilde{h}(t)e^{-t\mathbf{L}_n^\epsilon}\mathbf{f}\text{d}t = \mathbf{h}(\mathbf{L}_n^\epsilon)\mathbf{f}, \quad \mathbf{g}, \mathbf{f} \in \mathbb{R}^n.$$

- The frequency representation with the spectrum of $\mathbf{L}_n^\epsilon$ – $\{\lambda_{i,n}^\epsilon, \phi_{i,n}^\epsilon\}_{i=1}^n$

$$\mathbf{g} = \sum_{i=1}^n \hat{h}(\lambda_{i,n}^\epsilon)\langle \mathbf{f}, \phi_{i,n}^\epsilon\rangle_{L^2(\mathbf{G}_n)}\phi_{i,n}^\epsilon$$

Graph Laplacian spectrum approximation of the LB operator

Proposition (Difference of Laplacian operators)

Let $\mathcal{M} \subset \mathbb{R}^N$ be equipped with LB operator $\mathcal{L}$ whose spectrum is given by $\{\lambda_i, \phi_i\}_{i=1}^\infty$, and assume $\phi_i \in C(\mathcal{M})$. Let $\mathbf{G}_n$ be the discrete graph sampled u.i.d. from $\mathcal{M}$, with edge weights set with $\epsilon = \epsilon(n) > n^{-1/(d+4)}$ and graph Laplacian $\mathbf{L}_n^\epsilon$. It holds with probability at least $1 - \delta$ that

$$|\mathbf{L}_n^\epsilon\phi_i(x) - \mathcal{L}\phi_i(x)| \leq \left(C_1\sqrt{\frac{\ln(1/\delta)}{2n}} + C_2\sqrt{\epsilon}\right)\lambda_i^{\frac{d+2}{4}}$$

Graph Laplacian approximation of the LB operator

Proposition (Difference of Spectrum)

Let $\mathcal{M} \subset \mathbb{R}^N$ be equipped with LB operator $\mathcal{L}$ whose spectrum is given by $\{\lambda_i, \phi_i\}_{i=1}^\infty$. Let $\mathbf{G}_n$ be the discrete graph sampled u.i.d. from $\mathcal{M}$, with edge weights set with $\epsilon = \epsilon(n) > n^{-1/(d+4)}$ and graph Laplacian $\mathbf{L}_n^\epsilon$ with spectrum $\{\lambda_{i,n}^\epsilon, \phi_{i,n}^\epsilon\}_{i=1}^n$. Fix $K \in \mathbb{N}$ and assume $\epsilon = \epsilon(n) > n^{-1/(d+4)}$. Then, with probability at least $1 - 2e^{-n}$, we have

$$|\lambda_{i,n}^\epsilon - \lambda_i| \leq \Omega_1\sqrt{\epsilon}, \quad \|a_i\phi_{i,n}^\epsilon - \phi_i\| \leq \Omega_2\sqrt{\epsilon}$$

with $a_i = \{-1, 1\}$ for all $i < K$.

Frequency Dependent Filters

Definition (α-separated spectrum)

The α-separated spectrum of a LB operator $\mathcal{L}$ is defined as the partition $\Lambda_1(\alpha) \cup \ldots \cup \Lambda_N(\alpha)$ such that all $\lambda_i \in \Lambda_k(\alpha)$ and $\lambda_j \in \Lambda_l(\alpha)$, $k \neq l$, satisfy

$$|\lambda_i - \lambda_j| > \alpha.$$

Definition (α-FDT filter)

The frequency response of α-frequency Difference threshold (α-FDT) filter $\mathbf{h}(\mathcal{L})$ satisfies

$$|\hat{h}(\lambda_i) - \hat{h}(\lambda_j)| \leq \delta_k, \text{ for all } \lambda_i, \lambda_j \in \Lambda_k(\alpha)$$

with $\delta_k \leq \delta$ for $k = 1, 2 \ldots, n$.

Convergence of Graph Filtering

Theorem (Convergence of Graph Filtering)

Let $\mathbf{G}_n$ be a discrete graph sampled from manifold $\mathcal{M}$. Let $\mathbf{h}(\cdot)$ be the convolutional filter parameterized by the discrete graph Laplacian operator $\mathbf{L}_n^\epsilon$ or the LB operator $\mathcal{L}$. If it holds that

(H1) Weight values in $\mathbf{G}_n$ are set with $\epsilon = \epsilon(n) \geq n^{-1/(d+4)}$

(H2) Frequency response of $\mathbf{h}$ is A_h Lipschitz continuous and non-amplifying

(H3) Filter $\mathbf{h}$ is α-FDT with $\alpha^2 \gg \epsilon$ and $\delta = \Omega_2'\sqrt{\epsilon}/\alpha$, then the following holds in probability at least $1 - 2n^{-2}$

$$\|\mathbf{h}(\mathbf{L}_n^\epsilon)\mathbf{P}_n f - \mathbf{P}_n\mathbf{h}(\mathcal{L})f\|_{L^2(\mathbf{G}_n)} \leq \left(\frac{N\Omega_2'}{\alpha} + A_h\Omega_1\right)\sqrt{\epsilon} + C_{gc}\sqrt{\frac{\log n}{n}}$$

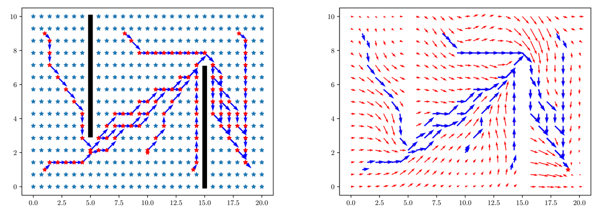
- Observe the trade-off between the approximation and discriminability
- Transferability can be derived based on this non-asymptotic error bound

Navigation Control

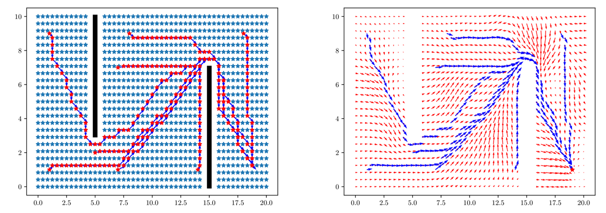
- We evaluate the graph filtering approximation with navigation control

Cervino, J. et al, Learning globally smooth functions on manifold, arXiv:2210.00301, 2022

- We predict the potential direction leading to the goal point based on generated trajectories



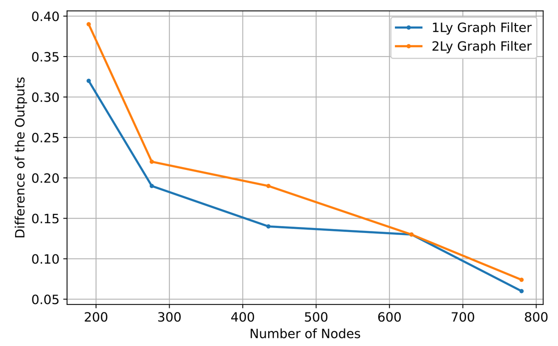
Experiments on graphs $n = 413$



Experiments on graphs $n = 1117$

Navigation Control Convergence and Transferability Results

- We train the graph filters on small graphs and plot the output differences
- We verify the transferability by testing the trained graph filters on $n = 1225$



	1Ly GF	2Ly GF
$n = 435$	0.74	0.74
$n = 630$	0.79	0.8
$n = 780$	0.81	0.8
$n = 1225$	0.82	0.83

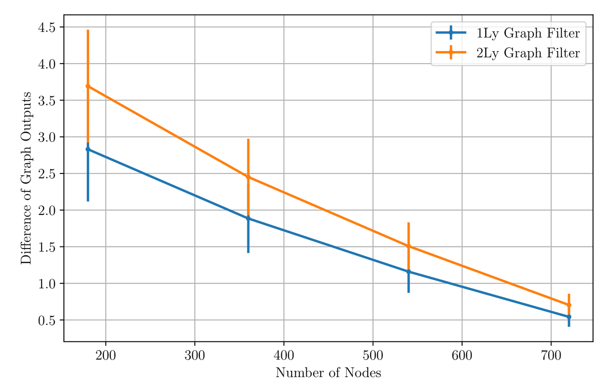
Table: Successful rates

Pointcloud Model Classification Result

- We evaluate the graph filtering approximation with ModelNet10 classification

Wu, Z. et al,3d shapenets: A deep representation for volumetric shapes,IEEE CVPR 2015

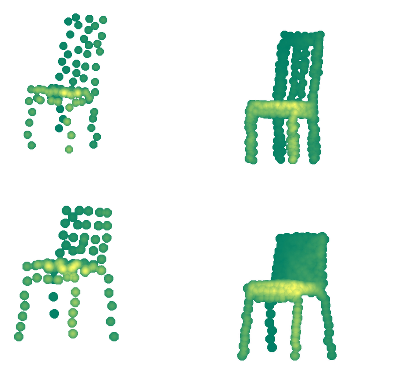
- Plot the graph output differences between trained graphs and a large graph



- Graph filters can converge to manifold filters as more points are sampled

Pointcloud Model Classification Transferability Result

- We verify the transferability by testing the trained graph filters on $n = 1000$



	1Ly Graph Filter	2Ly Graph Filter
$n = 300$	$21.15\% \pm 3.48\%$	$19.25\% \pm 3.47\%$
$n = 500$	$18.09\% \pm 6.28\%$	$17.80\% \pm 7.52\%$
$n = 700$	$17.31\% \pm 6.59\%$	$14.16\% \pm 5.93\%$
$n = 900$	$15.58\% \pm 4.54\%$	$12.21\% \pm 5.74\%$

Table: Error rates testing on $n = 1000$

- Transferability allows trained graph filters directly applied to a large graph